Symbolic-Numeric Factorization of Linear Differential Operators

Frédéric Chyzak [1] **Alexandre Goyer** [1] Marc Mezzarobba [2]

[1]Inria Saclay – Île-de-France, France

[2]LIX, CNRS, Computer Science Laboratory of École Polytechnique, France

Journées Nationales de Calcul Formel, CIRM, France (Luminy)

February 28, 2022

The problem

Object of study: linear differential operator

$$P = a_r\partial^r + a_{r-1}\partial^{r-1} + \cdots + a_1\partial + a_0 \in \mathbb{K}(x)\langle\partial\rangle$$

for a number field $\mathbb{K} \subset \mathbb{C}$, $a_i \in \mathbb{K}(x)$, $a_r \neq 0$, and $\partial = d/dx$

Commutation rule: $\partial x = x\partial + 1$ Leibniz: $(xf)' = xf' + f$

Goal

Find $L, R \in \overline{\mathbb{K}}(x)\langle\partial\rangle$ of order > 0 such that $P = LR$

Example:

$$x\partial^2 + (-4x^3 + 5x)\partial + 4x^2 - 5 = (\partial - 4x^2 + 5)(x\partial - 1)$$

History: algorithms of factorization

1894 **Beke**
hyperexponential solutions
+ exterior power method

1996 **Singer**
eigenring method

1997 **van Hoeij**
local-to-global method
➤ Maple DEtools package

2004 **Cluzeau**, **van Hoeij**
modular approach

2007 **van der Hoeven**
symbolic-numeric approach
➤ SageMath package (by G., 2021)

2013 **Johansson**, **Kauers**, **Mezzarobba**
hyperexponential solutions by symbolic-numeric approach

2014 **Llorente**
Liouvillian solutions by symbolic-numeric approach

Improvements of Beke's algorithm

1989 **Schwarz**

1990 **Grigor'ev**
(complexity analysis)

1994 **Bronstein**

1996 **Tsarev**

Solution space

Definition: $\{\text{singularities of } P\} = \bigcup_{i=0}^{r-1} \{\text{poles of } \frac{a_i}{a_r}\}$

Assumption: 0 is not a singularity of P (throughout this talk)

Cauchy-type theorem:
For each $0 \leqslant i \leqslant r-1$, there is a unique $f_i \in \mathbb{C}[[x]]$ such that
$P(f_i) = 0$ and $f_i^{(j)}(0) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ for each $0 \leqslant j \leqslant r-1$.

$$\mathrm{Sol}(P) := \mathrm{Span}_{\mathbb{C}}(f_0, \ldots, f_{r-1})$$

Identification:
$\mathrm{Sol}(P) \ni f \longleftrightarrow \mathrm{coord}(f) \in \mathbb{C}^r$ (coordinates in the basis $(f_i)_i$)

Differential Galois group

e.g. [Singer–van der Put, 2003]

$E := \mathbb{C}(x)(f_0, \ldots, f_{r-1}, f_0', \ldots, f_{r-1}', \ldots, f_0^{(r-1)}, \ldots, f_{r-1}^{(r-1)})$

$\mathrm{Gal}_{\mathrm{diff}}(P) := \{ \sigma \in \mathrm{Aut}\,(E/\mathbb{C}(x)) \mid \forall f \in E,\ \sigma(f') = \sigma(f)' \}$

- left action of $\mathrm{Gal}_{\mathrm{diff}}(P)$ on $\mathrm{Sol}(P)$

Theorem: $\mathrm{Gal}_{\mathrm{diff}}(P)$ is a linear algebraic group.

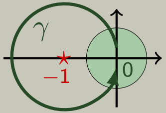
Faithful representation: $\psi : \sigma \mapsto M \in \mathbb{C}^{r\times r}$ such that

$$\mathrm{coord}(\sigma(f)) = M \times \mathrm{coord}(f)$$

- left action of $\mathcal{G} := \psi\,(\mathrm{Gal}_{\mathrm{diff}}(P))$ on $\mathbb{C}^r$

Example/Definition

$Q = (x+1)\partial^2 + \partial$ admits $f_0 := 1$, $f_1 := \log(1+x)$ as basis of solutions



$$af_0 + bf_1 \xrightarrow{\text{analytic continuation along } \gamma} (a + 2i\pi b)f_0 + bf_1$$

$$\begin{pmatrix} a + 2i\pi b \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

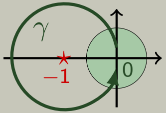
local monodromy matrix around -1

$\mathbb{C}^{r\times r} \supset \mathcal{M} :=$ group generated by all the local monodromy matrices of P

Theorem [Schlesinger, 1885] If P is regular then $\overline{\mathcal{M}} = \mathcal{G}$ (Zariski topology).

Example/Definition

$Q = (x+1)\partial^2 + \partial$ admits $f_0 := 1$, $f_1 := \log(1+x)$ as basis of solutions



$$af_0 + bf_1 \xrightarrow{\text{analytic continuation along } \gamma} (a + 2i\pi b)f_0 + bf_1$$

$$\begin{pmatrix} a + 2i\pi b \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2i\pi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

local monodromy matrix around -1

$\mathbb{C}^{r \times r} \supset \mathcal{M} :=$ group generated by all the local monodromy matrices of P

Theorem [Schlesinger, 1885] If P is regular then $\overline{\mathcal{M}} = \mathcal{G}$ (Zariski topology).

- How to check the regularity? $\rightarrow$ Fuchs' Criterion [Fuchs, 1866]
- What if P is not regular?
  $\rightarrow$ add exponential matrices and Stokes' matrices [Ramis, 1985]
- rigorous arbitrary-precision computation of monodromy matrices is available in SageMath [Mezzarobba, 2016]

Module point of view

e.g. [Singer–van der Put, 2003]

$\mathcal{A} := \mathbb{C}[\mathcal{M}] \subset \mathbb{C}^{r\times r}$ (group algebra)

Proposition: There is a one-to-one correspondance:

right-hand factor R of P $\xleftrightarrow[V \simeq \mathrm{Sol}(R)]{}$ $\mathcal{A}$-submodule V of $\mathbb{C}^r$

Strategy for finding a right-hand factor [van der Hoeven, 2007]

1. Compute approximations of the local monodromy matrices
2. Search for an approximate nontrivial proper $\mathcal{A}$-submodule V
3. If no such V exists then
4. Exploit errors bounds to ensure the irreducibility
5. Else guess a symbolic $R \in \overline{\mathbb{K}}(x)\langle\partial\rangle$ from V and check it

Finding a submodule

Assume that P admits a nontrivial factorization $P = LR$.

$\tilde{\mathcal{A}} := T^{-1}\mathcal{A}T$ where $T :=$ transition matrix from $(f_0, \ldots, f_{r-1})$ to a basis adapted to a decomposition $\mathrm{Sol}(P) = \mathrm{Sol}(R) \oplus S$

Note: Any $M \in \tilde{\mathcal{A}}$ have a bloc-triangular form $\left(\begin{array}{c|c} M_R & * \\ \hline 0 & M_L \end{array}\right)$

If there is $M \in \tilde{\mathcal{A}}$ with a simple eigenvalue λ, then

$$\begin{cases} \lambda \text{ eigenvalue of } M_R & \Rightarrow \quad \tilde{\mathcal{A}}v \subsetneq \mathbb{C}^{r\times 1} \\ \lambda \text{ eigenvalue of } M_L & \Rightarrow \quad w\tilde{\mathcal{A}} \subsetneq \mathbb{C}^{1\times r} \end{cases}$$

where $v \in \mathbb{C}^{r\times 1}$, $w \in \mathbb{C}^{1\times r}$ satisfy $Mv = \lambda v$, $wM = \lambda w$.

Norton's criterion (variant)

e.g. [Holt–Rees, 1994]

Theorem: Given $M \in \mathcal{A}$, a simple eigenvalue λ, and nonzero $v \in \mathbb{C}^{r\times 1}$, $w \in \mathbb{C}^{1\times r}$ satisfying $Mv = \lambda v$, $wM = \lambda w$, the following statements are equivalent:

(1) the left $\mathcal{A}$-module $\mathbb{C}^{r\times 1}$ is irreducible,

(2) $\mathcal{A}v = \mathbb{C}^{r\times 1}$ and $w\mathcal{A} = \mathbb{C}^{1\times r}$,

(3) the right $\mathcal{A}$-module $\mathbb{C}^{1\times r}$ is irreducible.

Note: right action of $\mathcal{A}$ on $\mathbb{C}^{1\times r} \longleftrightarrow$ monodromy action on $\mathrm{Sol}(P^*)$
$P^* :=$ image of P by the anti-endomorphism of $\mathbb{K}(x)\langle\partial\rangle$ mapping ∂ to $-\partial$
(so that $(LR)^* = R^*L^*$)

Reconstructing a right-hand factor from a seed vector v

Proposition: $\mathcal{A}v \simeq \mathrm{Sol}(P_f)$ where $P_f :=$ minimal annihilator of f

Hermite–Padé approximants

Given $f, f', \ldots, f^{(r-1)} \in \overline{\mathbb{K}}[[x]]$, find $p_0, \ldots, p_{r-1} \in \overline{\mathbb{K}}[x]$ of degree $\leq d$ such that $\sum_{i=0}^{r-1} p_i f^{(i)} = o(x^\sigma)$. $(\sigma < r(d+1) \Rightarrow$ nonzero solutions$)$

- if we find a candidate right-hand factor (using LLL algorithm to guess coefficients in $\overline{\mathbb{K}}$), we can check it by Euclidean division in $\overline{\mathbb{K}}(x)\langle\partial\rangle$.

Reconstructing a right-hand factor from a seed vector v

Proposition: $\mathcal{A}v \simeq \mathrm{Sol}(P_f)$ where $P_f :=$ minimal annihilator of f

Hermite–Padé approximants

Given $f, f', \ldots, f^{(r-1)} \in \overline{\mathbb{K}}[[x]]$, find $p_0, \ldots, p_{r-1} \in \overline{\mathbb{K}}[x]$ of degree $\leq d$ such that $\sum_{i=0}^{r-1} p_i f^{(i)} = o(x^\sigma)$. $(\sigma < r(d+1) \Rightarrow$ nonzero solutions$)$

- if we find a candidate right-hand factor (using LLL algorithm to guess coefficients in $\overline{\mathbb{K}}$), we can check it by Euclidean division in $\overline{\mathbb{K}}(x)\langle\partial\rangle$.
- irreducible case $\rightarrow$ computations at a large order of truncation ($\sigma > rB$ where B is a degree bound for right-hand factors)
- $\mathcal{A}v = \mathbb{C}^{r\times 1}$ can be ensured without HP computation (but a complete set of generators of $\mathcal{A}$ is needed)

Spin-up algorithm

1. Start with $V = \mathbb{C}v$ [represented by a basis]
2. Add to V all the vectors Mu for $M \in \mathcal{A}$, $u \in V$ until saturation
3. Return V

Algorithm RightFactor(P) [Chyzak–G.–Mezzarobba, 2022]

Input: $P \in \mathbb{K}(x)\langle\partial\rangle$ of order r
Output: `'Irreducible'` or a nontrivial proper right-hand factor of P

1. Initialize the working precision p and the order of truncation σ
2. While a complete set of generators of $\mathcal{A}$ has not been computed:
   - 2.a. Compute a new local monodromy matrix
   - 2.b. Choose randomly a combination M of the computed generators
   - 2.c. Try Norton's criterion or another method depending on the spectrum of M
   - 2.d. If the previous step succeeds then return the result
3. Increase p and σ, and go back to line 2

Lemma [Eberly, 1991]
The set $\{M \in \mathcal{A}$ without simple eigenvalue$\}$ is an algebraic subset of $\mathcal{A}$.

Note: Local-to-global method $\simeq$ 1st iteration of While loop

Experiments

operator	order, degree in x	classic (♒)	new (♋)	
rand (♏)	4, 20	6.6s	8.3s	local-to-global method applies
	6, 30	980s (!)	180s	
rand×rand (♏) (different factors)	4, 20	13s	2.7s	
	6, 30	240s (!)	25s	
rand (♏)	4, 20	1400s	26s	local-to-global method does not apply
	6, 30	2900s (!)	270s	
rand×rand (♏) (different factors)	4, 20	1700s	7.2s	
	6, 30	14000s (!)(!!)	130s	
fcc5 (♈)	6, 17	62s	2.0s	
fcc6 (♈)	8, 43	>36000s	30s	

(♒) command `DFactor` of the Maple package `DEtools` (author: van Hoeij)

(♋) command `.factor()` of my branch of `ore_algebra` SageMath package, at https://github.com/a-goyer/ore_algebra/tree/facto under the GNU GPL

(♏) random irreducible operators, created following [Ince, 1926]

(♈) irreducible operators at http://koutschan.de/data/fcc1/ (probabilistic walks)

(!) Warning message: "factorization of ... may be incomplete" (!!) Factorization is incomplete

Overview

Contributions

- Package for symbolic-numeric factorization, validation
- Link between local-to-global approach and Norton's criterion
- Incremental hybrid algorithm + implementation in progress

Remaining work

- Further timing comparisons between different approaches
- Complexity analysis
- Extension to non-regular case
- Auxiliary questions:
  - Exponential/Liouvillian solutions
  - Decide whether there is a basis of algebraic solutions
  - The differential Galois group or its Lie algebra

Derek F. Holt and Sarah Rees.
Testing modules for irreducibility.
Journal of the Australian Mathematical Society, 57(1):1–16, 1994.

Edward Lindsay Ince.
Ordinary Differential Equations.
Dover Publications, New York, 1926.

Claude Mitschi and David Sauzin.
Divergent Series, Summability and Resurgence II. Monodromy and Resurgence.
Springer, 2016.

Joris van der Hoeven.
Around the numeric-symbolic computation of differential Galois groups.
Journal of Symbolic Computation, pages 236–264, 2007.

Marius van der Put and Michael F. Singer.
Galois Theory of Linear Differential Equations.
Springer-Verlag Berlin Heidelberg, 2003.

Mark van Hoeij.
Factorization of Linear Differential Operators.
PhD thesis, Catholic University of Nijmegen, 1996.